

# On the first solution of a long standing problem: uniqueness of the phaseless quantum inverse scattering problem in 3-d

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## Abstract

After publishing his recent paper in SIAM J. Appl. Math, 74, 392-410, 2014 the author has realized that actually he has addressed in that paper, for the first time, a long standing open question being unaware about this. This question is about the uniqueness of a 3-d inverse scattering problem without the phase information. Thus, it makes sense in the current paper to explicitly make the latter statement and to formulate corresponding uniqueness theorems.

## 1 Introduction

In quantum scattering one is measuring the differential scattering cross section. In the frequency domain, this means that the modulus of the scattering complex valued wave field is measured. However, the phase is not measured. On the other hand, the entire theory of the inverse quantum scattering problem is based on the assumption that both modulus and phase are measured outside of a compact support of a scatterer. The latter was noticed in Chapter 10 of the well known book of K. Chadan and P.C. Sabatier published in 1977 [2]. In particular, Chadan and Sabatier state in introduction to Chapter 10 “*In typical situations, the scattering phase remains deeply involved in the formulas. Therefore, mathematical ways of constructing the scattering amplitude from the cross sections will be of interest for years*”.

Recently the author has proved, for the first time, uniqueness theorems for the phaseless inverse scattering problem (PISP) in the 3-d case [6]. However, only after the paper [6] was published, the author has realized that [6] is the first publication where the long standing problem posed in [2] is addressed, at least partially. Thus, the goal of the current paper is to draw attention to this fact. We formulate here two of our four theorems of [6] and briefly

outline ideas of their proofs. The idea of [6] was extended in [7] to the case of a 3-d phaseless inverse problem for the acoustic equation. While [6] is about the 3-d case, the 1-d phaseless case was first considered by the author and Sacks in [8].

## 2 Statements of Problems

Below  $C^{s+\alpha}$  are Hölder spaces, where  $s \geq 0$  is an integer and  $\alpha \in (0, 1)$ . Let  $\Omega, G \subset \mathbb{R}^3$  be two bounded domains,  $\Omega \subset G$  and  $S = \partial G$  be a piecewise continuous boundary of  $G$ . We assume that

$$\text{dist}(S, \partial\Omega) \geq 2\varepsilon, \quad (1)$$

where the number  $\varepsilon > 0$  and  $\text{dist}(S, \partial\Omega)$  denotes the Hausdorff distance. Let the potential  $q(x), x \in \mathbb{R}^3$  be a real valued function such that

$$q(x) \in C^{m+\alpha}(\mathbb{R}^3), q(x) = 0 \text{ for } x \in \mathbb{R}^3 \setminus \Omega, q(x) \geq 0, \forall x \in \Omega. \quad (2)$$

Here either  $m = 2$  or  $m = 4$ , as will specified later. Let  $x_0 \in S$  be the position of the source. As the forward problem, we consider the following

$$\Delta_x u + k^2 u - q(x)u = -\delta(x - x_0), x \in \mathbb{R}^3, \quad (3)$$

$$u(x, x_0, k) = O\left(\frac{1}{|x - x_0|}\right), |x| \rightarrow \infty, \quad (4)$$

$$\sum_{j=1}^3 \frac{x_j - x_{j,0}}{|x - x_0|} \partial_{x_j} u(x, x_0, k) - iku(x, x_0, k) = o\left(\frac{1}{|x - x_0|}\right), |x| \rightarrow \infty. \quad (5)$$

Here the frequency  $k \in \mathbb{R}$  and conditions (4), (5) are valid for every fixed source position  $x_0$ . We now refer to works of Vainberg [10, 11] as well as to a classical result about elliptic PDEs of the book of Gilbarg and Trudinger [4]. Theorem 3.3 of the paper [10], Theorem 6 of Chapter 9 of the book [11] in combination with Theorem 6.17 of [4] guarantee that for each pair  $(k, x_0) \in \mathbb{R} \times \mathbb{R}^3$  there exists such a unique solution  $u(x, x_0, k)$  of the problem (3), (4), (5) that can be represented in the form

$$u(x, x_0, k) = u_0(x, x_0, k) + u_{sc}(x, x_0, k), \quad (6)$$

$$u_0 = \frac{\exp(ik|x - x_0|)}{4\pi|x - x_0|}, u_{sc} \in C^{m+2+\alpha}(\{|x - x_0| \geq \eta\}), \forall \eta > 0. \quad (7)$$

Here  $u_0$  is the incident spherical wave and  $u_{sc}$  the scattered wave.

**Phaseless Inverse Scattering Problem 1 (PISP1).** *Suppose that the function  $q(x)$  satisfies conditions (2), where  $m = 2$ . Assume that the following function  $f_1(x, x_0, k)$  is known*

$$f_1(x, x_0, k) = |u(x, x_0, k)|, \forall x_0 \in S, \forall x \in \{|x - x_0| < \varepsilon\}, x \neq x_0, \forall k \in (a, b), \quad (8)$$

where  $(a, b) \subset \mathbb{R}$  is an arbitrary interval. Determine the function  $q(x)$  for  $x \in \Omega$ .

This inverse problem is over-determined. Indeed, the function  $q(x)$  depends on three variables, whereas the function  $f_1(x, x_0, k)$  depends on six variables. However, it is well known that uniqueness theorems for inverse scattering problems in 3-d in the case when the  $\delta$ -function is considered as the source, are known only for the over-determined case, even if both modulus and phase of the scattering field are given outside of the scatterer, see, e.g. Chapter 6 of the book of Isakov [5].

Consider now the non over-determined case of the data resulting from a single measurement. In this case uniqueness theorems for coefficient inverse problems in  $n - d$ ,  $n \geq 2$  are known only if the  $\delta$ -function is replaced with a regular function which is non-zero in the entire domain of interest  $\overline{\Omega}$  where the coefficient is unknown. All these theorems were proven using the method, which was originated in the work of Bukhgeim and Klibanov [1], also see the recent survey of the author [9]. This method is based on Carleman estimates. Thus, we replace the  $\delta$ -function in (3) with the function  $g(x)$  such that

$$g \in C^4(\mathbb{R}^3), g(x) = 0 \text{ in } \mathbb{R}^3 \setminus G_1, g(x) \neq 0 \text{ in } \overline{G}, \quad (9)$$

where  $G_1 \subset \mathbb{R}^3$  is a bounded domain such that  $\Omega \subset G \subset G_1, \partial G \cap \partial G_1 = \emptyset$ . As it was mentioned in [6, 9], the function  $g(x)$  can be, for example an approximation for the  $\delta$ -function via a narrow Gaussian-like function, which is equivalent to the  $\delta$ -function from the Physics standpoint. As the forward problem, we consider the following

$$\Delta w + k^2 w - q(x) w = -g(x), x \in \mathbb{R}^3, \quad (10)$$

$$w(x, k) = O\left(\frac{1}{|x|}\right), |x| \rightarrow \infty, \quad (11)$$

$$\sum_{j=1}^3 \frac{x_j}{|x|} \partial_{x_j} w(x, k) - ikw(x, k) = o\left(\frac{1}{|x|}\right), |x| \rightarrow \infty. \quad (12)$$

The same results of [4, 10, 11] as ones mentioned above guarantee that for each  $k \in \mathbb{R}$  there exists unique solution  $v(x, k) \in C^{5+\alpha}(\mathbb{R}^3), \forall \alpha \in (0, 1)$  of the problem (10), (11), (12).

**Phaseless Inverse Scattering Problem 2 (PISP2).** Assume that in (2)  $m = 4$ , the function  $q(x)$  satisfying conditions (2) is unknown for  $x \in \Omega$  and known for  $x \in \mathbb{R}^3 \setminus \Omega$  and that the function  $g(x)$  satisfies conditions (9). Determine the function  $q(x)$  for  $x \in \Omega$  assuming that the following function  $f_2(x, k)$  is known

$$f_2(x, k) = |v(x, k)|, \forall x \in S, \forall k \in (a, b). \quad (13)$$

We now outline the main difficulty in addressing each of above inverse problems. We consider only the PISP1, since this difficulty is similar for the PISP2. For a fixed pair  $(x, x_0)$  denote  $p(k) = u(x, x_0, k), k \in \mathbb{R}$ . Thus, the modulus  $|p(k)|$  is known for all  $k \in (a, b)$ . For each number  $\gamma > 0$  denote

$$\mathbb{C}_\gamma = \{z \in \mathbb{C} : \text{Im } z > -\gamma\}, \mathbb{C}_+ = \{z \in \mathbb{C} : \text{Im } z > 0\}.$$

The function  $p(k)$  admits the analytic continuation from the real line  $\mathbb{R}$  in the half-plane  $\mathbb{C}_\gamma$  for a certain number  $\gamma > 0$  (see details below). For each  $z \in \mathbb{C}$  let  $\bar{z}$  be its complex conjugate. Since

$$|p(k)|^2 = p(k) \bar{p}(k), \forall k \in \mathbb{R}, \quad (14)$$

then the function  $|p(k)|^2$  is analytic for  $k \in \mathbb{R}$  as the function of the real variable  $k$ . Hence, the modulus  $|p(k)|$  is known for all  $k \in \mathbb{R}$ . Let  $\{a_j\}_{j=1}^n \subset \mathbb{C}_+$  be all zeros of the function  $p(k)$  in the upper half plane. Here and below each zero is counted as many times as its multiplicity is. Using an analog of classical Blaschke products [3], consider the function  $p_a(k)$ ,

$$\tilde{p}(k) = p(k) \prod_{j=1}^n \frac{k - \bar{a}_j}{k - a_j}. \quad (15)$$

The function  $\tilde{p}(k)$  is analytic in  $\mathbb{C}_\gamma$  and  $|p_a(k)| = |p(k)|, \forall k \in \mathbb{R}$ . Therefore, the central question is about finding of complex zeros. To do this, one needs to figure out how to combine the knowledge of  $|p(k)|$  for  $k \in \mathbb{R}$  with a linkage between the function  $p(k)$  and the original forward problem (3)-(5).

### 3 Uniqueness Theorems and Main Ideas of Proofs

**Theorem 1.** *The PISP1 has at most one solution.*

**Theorem 2.** *The PISP2 has at most one solution.*

We now outline main ideas of proofs of these theorems. We start from Theorem 1. Consider the following hyperbolic Cauchy problem

$$v_{tt} = \Delta_x v - q(x) v, (x, t) \in \mathbb{R}^3 \times (0, \infty), \quad (16)$$

$$v(x, 0) = 0, v_t(x, 0) = \delta(x - x_0). \quad (17)$$

Let  $\Phi \subset \mathbb{R}^3$  be an arbitrary bounded domain. Using lemma 6 of Chapter 10 of [11] as well as Remark 3 after that lemma, one can proof that for each fixed source position  $x_0 \in \mathbb{R}^3$  the function  $v(x, x_0, t)$  decays exponentially with respect to  $t \rightarrow \infty$  uniformly for all  $x \in \Phi, x \neq x_0$  together with its derivatives involved in (16). One can derive from the latter that

$$u(x, x_0, k) = \int_0^\infty v(x, x_0, t) e^{ikt} dt = \mathcal{F}(v), \forall x, x_0 \in \mathbb{R}^3, x \neq x_0, \forall k \in \mathbb{R}, \quad (18)$$

where  $\mathcal{F}$  is the operator of the Fourier transform (18). Hence, for every pair  $x, x_0 \in \mathbb{R}^3, x \neq x_0$  there exists a number  $\gamma = \gamma(x, x_0) > 0$  such that the function  $u(x, x_0, k)$  can be analytically continued with respect to  $k$  in the half-plane  $\mathbb{C}_\gamma$ . Next, using (16)-(18), we derive the behavior of the function  $u(x, x_0, k)$  for  $|k| \rightarrow \infty, k \in \mathbb{C}_+$  and conclude that this function has at most finite number of zeros in  $\mathbb{C}_+$ , for each above pair  $x, x_0$ .

Now the turn is for the most difficult part of the proof: we should prove that these zeros can be uniquely determined using the function  $f_1(x, x_0, k)$  in (8). Assume that there exist

two potentials  $q_1(x)$  and  $q_2(x)$  satisfying conditions (2) with  $m = 2$  and producing the same function  $f_1(x, x_0, k)$ . Let  $u_1(x, x_0, k)$  and  $u_2(x, x_0, k)$  be solutions of the problem (3)- (5) with functions  $q_1(x)$  and  $q_2(x)$  respectively. Fix two arbitrary points  $x_0 \in S$  and  $x \in \{y : |y - x_0| < \varepsilon\}, x \neq x_0$ . Consider corresponding functions  $h_1(k) = u_1(x, x_0, k)$  and  $h_2(k) = u_2(x, x_0, k)$ . Then both these functions are analytic in  $\mathbb{C}_\gamma$  and  $|h_1(k)| = |h_2(k)|, \forall k \in (a, b)$ . First, using (14), we show that

$$|h_1(k)| = |h_2(k)|, \forall k \in \mathbb{R} \quad (19)$$

and that real zeros of functions  $h_1(k)$  and  $h_2(k)$  coincide.

As to the more difficult case of complex zeros in  $\mathbb{C}_+$ , we consider analogs Blaschke products (15). Let  $\{a_j\}_{j=1}^{n_1} \subset \mathbb{C}_+$  and  $\{b_j\}_{j=1}^{n_2} \subset \mathbb{C}_+$  be all zeros of functions  $h_1(k)$  and  $h_2(k)$  respectively in the upper half plane. Then (19) implies that

$$h_1(k) + h_1(k) \left( \prod_{j=1}^m \frac{k - b_j}{k - \bar{b}_j} - 1 \right) = h_2(k) + h_2(k) \left( \prod_{j=1}^n \frac{k - a_j}{k - \bar{a}_j} - 1 \right). \quad (20)$$

Next, we use the partial fraction expansion in (20) and the following formula which can be proved via a straightforward computation for every integer  $s \geq 1$

$$\mathcal{F}^{-1} \left( \frac{1}{(k - \bar{d})^s} \right) = H(t) \frac{(-i)^s}{(s-1)!} t^{s-1} \exp(-i\bar{d}t), \quad \forall d \in \mathbb{C}_+, \quad (21)$$

where  $H(t)$  is the Heaviside function,  $H(t) = 1$  for  $t > 0$  and  $H(t) = 0$  for  $t < 0$ . Let  $v_j(x, x_0, t) = \mathcal{F}^{-1}(h_j), j = 1, 2$ . Since  $q_1(x) = q_2(x) = 0$  outside of the domain  $\Omega, x \in \{|x - x_0| < \varepsilon\}$  and by (1)  $\{|x - x_0| < \varepsilon\} \cap \bar{\Omega} = \emptyset$ , then (16) and (17) imply that  $v_1(x, x_0, t) = v_2(x, x_0, t)$  for  $t \in (0, |x - x_0| + \varepsilon)$ . Hence, applying the operator  $\mathcal{F}^{-1}$  to both sides of (20), using (21) and the convolution theorem for the Fourier transform, we obtain that there exists a function  $\lambda(t)$  which depends on above zeros  $\{a_j\}_{j=1}^{n_1}, \{b_j\}_{j=1}^{n_2}$  such that it satisfies a homogeneous Volterra integral equation of the second kind for sufficiently small values of  $t > 0$ . Hence,  $\lambda(t) = 0$  for sufficiently small  $t > 0$ . Next, since  $\lambda(t)$  is an analytic function of the real variable  $t$  for  $t > 0$ , then  $\lambda(t) = 0$  for all  $t > 0$ . This leads to the conclusion that  $\{a_j\}_{j=1}^{n_1} = \{b_j\}_{j=1}^{n_2}$  and  $n_1 = n_2$ . This, in turn implies that even if the function  $f_1(x, x_0, k)$  in (8) is known for only a single pair  $x_0 \in S, x \in \{|x - x_0| < \varepsilon\}, x \neq x_0$  and for  $k \in (a, b)$ , then values of  $u(x, x_0, k)$  are still uniquely determined for the same pair  $x, x_0$  and for all  $k \in \mathbb{R}$ .

The final step is to prove that the potential  $q(x)$  is uniquely determined. To do this, we show that

$$\int_{L(x, x_0)} (q_1 - q_2)(x) ds = 0, \quad \forall x, x_0 \in S, x \neq x_0.$$

Finally the uniqueness of the Radon transform implies that  $q_1(x) \equiv q_2(x)$ .

The proof of Theorem 2 is similar with only one difference. Namely, on the final step of the proof we use theorem 3.2 of [9] instead of the uniqueness of the Radon transform.

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## References

- [1] A.L. Bukhgeim and M.V. Klibanov, Uniqueness in the large of a class of multidimensional inverse problems, *Soviet Math. Doklady*, 17, 244-247, 1981.
- [2] K. Chadan and P.C. Sabatier, *Inverse Problems in Quantum Scattering Theory*, Springer-Verlag, New York, 1977.
- [3] P. Colwell, *Blaschke Products: Bounded Analytic Functions*, University of Michigan Press, Ann Arbor, 1985.
- [4] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, New York, 1984.
- [5] V. Isakov, *Inverse Problems for Partial Differential Equations*, Second Edition, Springer, New York, 2006.
- [6] M.V. Klibanov, Phaseless inverse scattering problems in three dimensions, *SIAM J. Appl. Math.*, 74, 392-410, 2014.
- [7] M.V. Klibanov, Uniqueness of two phaseless non-overdetermined inverse acoustics problems in 3-d, accepted for publication in *Applicable Analysis*, available online of this journal <http://dx.doi.org/10.1080/00036811.2013.818136>.
- [8] M.V. Klibanov and P.E. Sacks, Phaseless inverse scattering and the phase problem in optics, *J. Math. Physics*, 33, 3813-3821, 1992.
- [9] M. V. Klibanov, Carleman estimates for global uniqueness, stability and numerical methods for coefficient inverse problems, *J. Inverse and Ill-Posed Problems*, 21, 477-560, 2013.
- [10] B.R. Vainberg, Principles of radiation, limiting absorption and limiting amplitude in the general theory of partial differential equations, *Russian Math. Surveys*, 21, 115-193, 1966.
- [11] B.R. Vainberg, *Asymptotic Methods in Equations of Mathematical Physics*, Gordon and Breach Science Publishers, New York, 1989.